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On Fixed Points of Conformal Pseudogroups

Xavier Gómez-Mont and Bruno Wirtz

Abstract. We show in this paper Theorem 2 that if (H, H_1) is a pseudogroup generated by a finite number H_1 of germs of conformal diffeomorphisms of $\mathbb{C}$ defined on a sufficiently small disc D, which is not linearizable and such that the linear group $(L, H_1) = \{g'(0)/g \in (H, H_1)\} \subset \mathbb{C}^*$ is dense in $\mathbb{C}^*$, then the set of fixed points of the pseudogroup (H, H_1) is dense in D. This implies the abundance of distinct homotopy classes of loops in leaves of foliations defined in $\mathbb{C}^2$ by generic polynomial vector fields as well as for germs of holomorphic vector fields in $\mathbb{C}^2$ beginning with generic jets, both of degree at least 2. These homotopy classes may be realized arbitrarily close to the line at infinity or to 0, respectively. This shows the genericity of polynomial vector fields with infinite Petrovsky-Landis genus ([**5**]).
The idea of the proof is very simple. If g is a non-linear conformal diffeomorphism with multiplier $\lambda = g'(0)$, then the map obtained by the composition of g and the linear map with multiplier λ^{-1} will have at 0 a fixed point of multiplicity at least 2. Since we may approximate λ^{-1} by elements h in the pseudogroup and the multiplicity of fixed points satisfy a law of conservation of number, we obtain that $h \circ g$ has fixed points close to 0. These fixed points appear as a by product of the relative non-linearity of the generators of the pseudogroup, since linearizable pseudogroups have 0 as an isolated fixed point. The fixed points obtained are not conjugate since they have distinct multipliers.
The main technical tool is the angular derivative introduced in [**8**]. It allows one to split the search for fixed points into two parts: One is to obtain a contraction and the other is to return arbitrarily close to the starting point without modifying the property of contraction. This is carried out since the angular derivative is multiplicative for compositions and is identically 1 for linear maps.

1. The angular derivative

The angular derivative Δ of a conformal diffeomorphism $g\colon \mathrm{Dom}(g) \to \mathbb{C}$, $g(0) = 0$ is defined for $z \in \mathrm{Dom}(g)$ as:

$$\Delta g(z) = z\frac{g'(z)}{g(z)}$$

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The real part of this derivative is $\frac{\partial}{\partial\theta}\operatorname{Arg}(g(z))$, where $z = re^{i\theta}$. An interesting property of this derivative is its compatibility with composition: if f and g are conformal diffeomorphisms with $f(0) = g(0) = 0$, and if z is in the domain of $g \circ f$ then

$$\Delta(g \circ f)(z) = (\Delta g)(f(z)) \cdot (\Delta f)(z)$$

We have $\Delta g(z) = 1 + O(z)$; and g is linear if and only if $\Delta g \equiv 1$.

If g_λ denotes the map obtained by composing g and the linear map with multiplier λ we have:

$$g_\lambda(z) := \lambda \cdot g(z), \quad \Delta g_\lambda(z) = \Delta g(z) \tag{1}$$

The multiplicity of a fixed point z_0 of g is the multiplicity as a root of $g(z) - z$ at z_0, i.e. it is simple or of multiplicity 1 if $g(z_0) = z_0$ and $g'(z_0) \neq 1$, and of multiplicity $m > 1$ if in local coordinates it can be written as

$$g(z) = z_0 + (z - z_0) + b_m(z - z_0)^m + O((z - z_0)^{m+1}), b_m \neq 0.$$

The multiplicity of fixed points satisfies a law of conservation of number, in the sense that under a small perturbation the local sum is preserved. This follows since we may compute this index as the intersection index of the graph of g with the diagonal.

We use the notation of discs in the complex plane:

$$D(z_0, r) := \{z \in \mathbb{C} / |z - z_0| < r\} \qquad D_\alpha := D(0, \alpha)$$

Lemma 1. *Let $g: D_\alpha \to g(D_\alpha) \subset \mathbb{C}$ be a non-linear conformal diffeomorphisms onto its image preserving 0 with power series expansion*

$$g(z) = a_1 z + a_m z^m + \dots, \qquad a_1 \neq 0, m \geq 2, a_m \neq 0 \tag{2}$$

and let $g_\lambda = \lambda \cdot g$. There exist $\epsilon > 0$ such that if $0 < \left|\lambda - a_1^{-1}\right| < \epsilon$ then g_λ has m simple fixed points near 0. If $z_0 \neq 0$ is a fixed point of g_λ, then $\lambda = \frac{z_0}{g(z_0)}$ and the multiplier of g_λ at z_0 is $\Delta_g(z_0)$.

Proof. Consider the map

$$G: D_\alpha \times \mathbb{C} \to \mathbb{C} \quad G(z, \lambda) = \lambda \cdot g(z)$$

It is parametrizing the family of maps $\mathcal{G} = \{g_\lambda\}$. The set of fixed points $Fix \subset D_\alpha \times \mathbb{C}$ of $\mathcal{G}$ is defined by the equation $G(z,\lambda) = z$ or

$$z\left[\frac{\lambda \cdot g(z)}{z} - 1\right] = 0,$$

and so Fix consists of the 0-section $0 \times \mathbb{C}$ and the graph of the function

$$\lambda(z) = \frac{z}{g(z)} = \frac{z}{a_1 z + a_m z^m + m \cdots} = a_1^{-1} - a_m a_1^{-2} z^{m-1} + \cdots$$

The above function $\lambda\colon D_\alpha \to \mathbb{C}^*$ has a branch point of order $m-1$ at $z = 0$ over a_1^{-1}. This implies that for values of λ_0 close to a_1^{-1} the equation $\lambda(z) = \lambda_0$ has $m-1$ solutions near 0, so there are $m-1$ fixed points of g_{λ_0} distinct from 0. To check that the fixed points are simple, let $z_0 \in D_\alpha$ and the multiplier of $g_{\lambda(z_0)}$ at z_0 is:

$$g'_{\lambda(z_0)}(z_0) = \frac{z_0}{g(z_0)} g'(z)|_{z=z_0} = \Delta g(z_0)$$

Since $\Delta g(0) = 1$ and the equation $\Delta g(z) = 1$ has a discrete set of solutions (due to the non-linearity of g), we conclude that there is a pointed neighbourhood of a_1^{-1} where Δ_g does not take the value 1, and hence the above $m-1$ fixed points of g_{λ_0} are simple for ϵ small. □

Using the expression (2), one obtains

$$\Delta g(z) = 1 + (m-1)\frac{a_m}{a_1} z^{m-1} + O(z^m)$$

Hence Δg has at 0 a ramification point of order $m-1$. If we denote by D_α^+ and D_α^- the subsets of D_α defined by $|\Delta g| > 1$ and $|\Delta g| < 1$ respectively, then each consists of $2m-2$ alternating curvilinear sectors, having each an angle at 0 of $\frac{\pi}{m-1}$. Fixed points of g_λ in D_α^+ correspond to repulsors and in D_α^- correspond to attractors. Since the $m-1$ non-zero fixed points of g_λ of Lemma 1 are solutions to $\lambda(z) = \lambda_0$, and the function λ has a branch point over 0 of order $m-1$ (so we may find conformal coordinates where it is w^{m-1}), we conclude that these fixed points have similar norms and its angles differ approximately by $\frac{2\pi}{m}$. Hence for most values of λ_0 near to a_0^{-1} these fixed points will lie in only one of D_α^+ or D_α^-; so that they will be simultaneously attractors or repellors. They will also tend to be indifferent simultaneously.

Another way to see the above sectors, is to consider $g_{a_1^{-1}}$, whose multiplier at 0 is 1. This map is topologically equivalent to a Fatou flower with $2m$ petals ([**7**]). Two adjacent semipetals will be expanding or contracting. Since the angular derivative is the same for all elements g_λ, Δg has the observed sectorial behaviour.

Example. Let $g_\lambda(z) = \lambda(z+z^2)\colon D_\alpha \to \mathbb{C}$. 0 is a fixed point of multiplicity 2 for g_1 and for $\lambda \neq 1$ we have besides 0 another fixed point z_f. 0 is an attractor if and only if $0 < |\lambda| < 1$ and z_f is an attractor if and only if $0 < |\lambda - 2| < 1$. Hence the possible configurations after a small perturbation of $\lambda_0 = 1$ is to have 2 repellors or 1 repellor and 1 attractor, that can either be 0 or z_f.

a) Let $\lambda_1 = 1.1 + 0.2i$ and $\alpha = 0.25$. For these parameter values, z_f is an attractor and 0 is a repellor. In figure 1 we can see the circle of radius α and its image. Then we take the iterates of a small circle centered at). Part of these images are attracted by z_f.

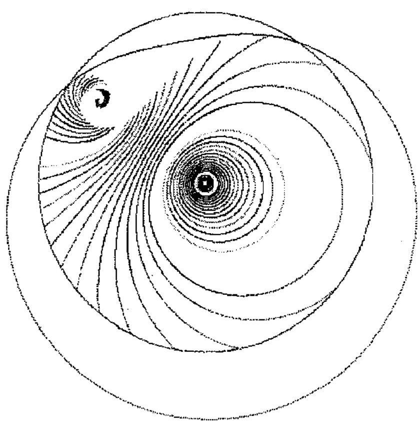

Figure 1: Some orbits for g_{λ_1}.

b) Let $\lambda_2 = 1 + 0.1i$ and $\alpha = 0.5$. For these parameter values, 0 and z_f are repellors. In figure 2 we can see the circle of radius α and its image. Then we take the iterates of a small circle centered at 0 and at z_f.

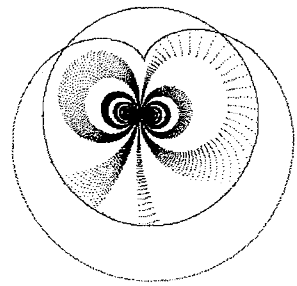

Figure 2: Some orbits for g_{λ_2}.

c) Let $\lambda_3 = 0.85 + 0.2i$ and $\alpha = 0.6$. For these parameter values, 0 is an attractor and z_f is a repellor. In figure 3 we can see the circle of radius α and its image. Then we take the iterates of a small circle centered at z_f. Part of these images are attracted by 0.

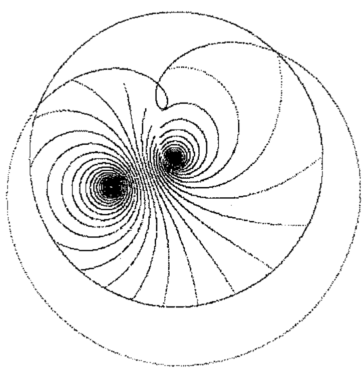

Figure 3: Some orbits for g_{λ_3}.

1.2. Conformal pseudogroups

Let (E, d) be a compact metric space and H_1 a finite collection of homeomorphisms between open sets of (E, d), stable by inversion, including Id_E. We define the domain of $f \circ g$ as $g^{-1}(\mathrm{Dom}(f) \cap \mathrm{Im}(g))$. The set H_n

is the collection of all compositions

$$f_n \circ f_{n-1} \circ \cdots \circ f_1; f_n, \ldots, f_1 \in H_1$$

whose domain of definition is the open subset of E formed by points $p \in E$ for which all intermediate points lie in the domain of the following map:

$$f_j \circ \cdots \circ f_1(p) \in \mathrm{Dom}(f_{j+1}) \quad \text{for} \quad j = 1, \ldots, n-1.$$

The pseudogroup (H, H_1) *generated by* H_1 is $\cup_{n \in \mathbb{N}} H_n$.

Given (E, d) and H_1 as above, let (H, H_1) be the pseudogroup generated by H_1. $p \in X$ is called a *fixed point of the pseudogroup* H if and only if there exists an element g of H such that $p \in \mathrm{Dom}(g)$, g is not the identity in a neighbourhood of p and $g(p) = p$.

We will consider the following type of conformal pseudogroups:

*) Let $\tilde{H}_1 = \{g_1, \ldots, g_k\}$ be a finite number of germs of conformal diffeomorphisms at $(\mathbb{C}, 0)$ such that it is stable under inverses: id, $g_j^{-1} \in \tilde{H}_1$. Assume that g_1 has a hyperbolic attractor at $0(|g_1'(0)| < 1)$, and let z be the coordinate that linearizes $g_1: g_1(z) = \lambda_1 z$. Choose $\alpha > 0$ such that all the functions in $H_1 := \{g_1, \ldots, g_k : D_\alpha \to \mathbb{C}\}$ are conformal diffeomorphisms defined on the disc D_α into $\mathbb{C}$.

The *linear group* associated to the above conformal pseudogroup (H, H_1) is the subgroup of $\mathbb{C}^*$ obtained by taking the derivatives of elements of (H, H_1) at 0:

$$(L, H_1) := \{g'(0) \in \mathbb{C} / g \in H\} = \{\lambda_1^{n_1} \ldots \lambda_k^{n_k} \in \mathbb{C}^* / \lambda_j = g_j'(0), n_j \in \mathbb{Z}\}.$$

Let (H, H_1) be a pseudogroup on D_α satisfying *). We may use the contraction $g_1(z) = \lambda_1 z$ as a microscope to magnify the behaviour near 0. The most important application is the fact that for an arbitrary element $g \in (H, H_1)$ the sequence of elements $\{g_2^{-n} \circ g \circ g_2^n\}_{n>0}$ of the pseudogroup converges uniformly in compact sets of

$$D_{\min\{\alpha, \frac{\alpha}{|g'(0)|}\}}$$

to the linear function with multiplier $g'(0)$. Hence we may find elements in (H, H_1) which approximate the elements of (L, H_1) in an arbitrary compact set in $D_{\min\{\alpha, \frac{\alpha}{|g'(0)|}\}}$.

Theorem 2. *Let (H, H_1) be a pseudogroup on D_α satisfying *), with some non-linear element in H_1, then:*

1) If the linear group (L, H_1) is not a discrete group of $\mathbb{C}^*$, then there are fixed points of the pseudogroup (H, H_1) arbitrarily close to 0.

2) If the linear group (L, H_1) is dense in $\mathbb{C}^*$, then the set of fixed points of (H, H_1) are dense in D_α.

Proof. 1) Let $g \in H_1$ be a non linear element. We may find $\lambda \in (L, H_1)$ arbitrarily close to $g'(0)^{-1}$ but not equal. By Lemma 1, the map λg will have simple fixed points distinct from 0. If $h: D_{\alpha-\epsilon} \to \mathbb{C}$ is a map that approximates the linear map with multiplier λ, then by continuity $h \circ g$ will also have a fixed point nearby. If $\tilde{g} \in (H, H_1)$ with $\tilde{g}'(0) = \lambda$ then the sequence $\{g_1^{-n} \circ \tilde{g} \circ g_1^n\}_n \subset (H, H_1)$ converge uniformly in compact sets of

$$D_{\min\{\alpha, \frac{\alpha}{|g'(0)|}\}}$$

to the linear map with multiplier λ. Hence

$$g_1^{-n} \circ \tilde{g} \circ g_1^n \circ g \in (H, H_1)$$

has a fixed point near to 0, but distinct from it.

2) Let $z_0 \in D_\alpha$. By density, we may approximate $\lambda = \frac{z_0}{g(z_0)}$ by elements in the linear group $\lambda_1^{n_1} \ldots \lambda_k^{n_k}$. Then we can consider the elements of H:

$$\{g_1^{-n} \circ g_1^{n_1} \cdots g_k^{n_k} \circ g_1^n\}_{n>0}$$

to approximate λz. By Lemma 1, we have that for a good approximation of λ and n large enough we obtain a periodic point near z_0 with multiplier near to $\Delta g(z_0)$. □

A pseudogroup (H, H_1) satisfying *) with a dense linear group (L, H_1) in $\mathbb{C}^*$ has the following properties:

1) The orbit of any point in $D_\alpha - \{0\}$ is dense in D_α ([**1**,**4**]).

2) The pseudogroup action is ergodic, in the sense that if $A \subset D_\alpha$ is a germ of a Borel measurable set at 0 invariant under the pseudogroup as a germ of a set at 0 then it has full or zero lebesgue measure [**4**,**3**].

3) The elements in H_1 are simultaneously linearizable if and only if the group of germs at 0 generated by H_1 is abelian. In the Abelian case,

the moduli of topologically equivalent pseudogroups has dimension 1, and in the non-Abelian case the pseudogroup is topologically rigid, in the sense that its moduli of topologically equivalent pseudogroups is 0 [**6**].

Theorem 2 exhibits another fundamental property of these pseudogroups: Abundance of fixed points.

Note that by conjugation we can move a fixed point, but all fixed points obtained in this manner will have the same multiplier at the corresponding fixed points. The fixed points that we obtained have multiplier the angular derivative Δg, and hence are in general non-conjugate fixed points.

As is well known, the monodromy group of the line at infinity in $\mathbb{C}^2$ for the global case, and of the exceptional divisor obtained by blowing up 0 at the local case, for generic polynomial or germs of vector fields of degree at least 2 ([**6,2,3**]) satisfies the hypothesis of Theorem 2, concluding the existence of non-trivial loops arbitrarily close to these punctured spheres.

Ilyashenko obtains examples of foliations with infinite number of limit cycles using a perturbation of a Hamiltonian vector field [**5**]. This paper gives a proof that for a generic polynomial vector field of degree at least 2, the Petrovsky-Landis genus is infinite.

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Xavier Gómez-Mont
CIMAT, AP 402
Guanajuato, 36000, México
GOMEZM @ BUZON.MAIN.CONACYT.MX

Bruno Wirtz
Dptm. Mathématiques
U. Bretagne Occidental, B. P. 809
29285 Brest Cedex
Brest, France
WIRTZ @ UNIV-BREST.FR